A REMARK ON CANTOR DERIVATIVE

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Abstract. It is shown that, modulo an equivalence relation induced by finite correspondences preserving Cantor rank, the class of topological spaces is an integral semi-ring on which the Cantor derivative is precisely a derivation.

The notions of limit point and derivated space have both been introduced by Georg Cantor in 1872 to derivate sets of convergence of trigonometric series. In [1] Cantor shows that the representation of a function as a trigonometric series is unique on a set minus some finite Cantor ranked set. It was as he considered points systems having infinite Cantor rank that he introduced transfinite induction. That was in 1880, three years before his set theory. Cantor's words are *Grenzpunkt* and *abgeleitete Punktmenge* for "limit point" and "derived space" respectively. He was already writing P' or $P^{(1)}$ for the first derivative of a set of points P. However, it does not seem at all that Cantor had in mind a Leibniz formula, but it is intriguing that the class of topological spaces can naturally be turned into a semi-ring where Cantor's derivative is actually a derivation.

All spaces considered in the sequel are topological spaces. A *correspondence* between two spaces X and Y is any relation $R \subset X \times Y$ such that the projections to X and Y are onto. We write R^{-1} for the inverse correspondence of R from Y to X. If O is an open set in X, we define $R(O)$ as $\{y \in Y : (x, y) \in R \text{ for some } x \in O\}$. The relation R is *continuous* if for every open set O in Y, the set $R^{-1}(O)$ is open in X. It is *open* if R^{-1} is continuous. It is a *n-to-m* correspondence if for all x, y in $X \times Y$, the set $R(\{x\})$ has cardinal at most m and $R^{-1}(\{y\})$ has cardinal at most n. A correspondence is *finite* if it is n-to-m for some non-zero integers n and m.

Let X be any topological space. We slightly modify the usual definitions to avoid the use of any separation axiom, and call a point *isolated* if it belongs to a finite open set. Otherwise, we say that it is a *limit*

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point. We shall write X' for the *derivative* of X, that is, the set of limit points with the induced topology, and define a descending chain of closed subsets X^α by setting, inductively

$$\begin{aligned}
&X^0 = X \\
&X^{\alpha+1} = \left(X^\alpha\right)' \text{ for a successor ordinal} \\
&X^\lambda = \bigcap_{\alpha<\lambda} X^\alpha \text{ for a limit ordinal } \lambda
\end{aligned}$$

We call *Cantor-Bendixson rank* of X, written $CB(X)$, the least ordinal α such that X^α is empty if such an ordinal exists, or ∞ otherwise. The rank of a point x is the supremum of the α such that $x \in X^\alpha$. A subset, or a point of X has *maximal* rank if it has the same Cantor-Bendixson rank as X. Otherwise, we say that it has *small* rank.

We call a *rough partition* of X, any covering of X by open sets having maximal rank and small ranked intersections. We define the *Cantor-Bendixson degree* $d(X)$ of X to be the supremum cardinal of the rough partitions of X. Open continuous finite correspondences do preserve the rank :

Lemma 1. *Let R be a n-to-m correspondence between two spaces X and Y.*

(*i*) *If R is open, $CB(X) \geq CB(Y)$.*
(*ii*) *If R is continuous, $CB(X) \leq CB(Y)$.*
(*iii*) *If R is continuous and open, then X and Y have the same rank and*

$$\frac{1}{m} \cdot d(Y) \leq d(X) \leq n \cdot d(Y)$$

Proof. (*i*) Let y be a limit point in Y and x in $R^{-1}(\{y\})$. For every neighbourhood O of x, the image $R(O)$ is an infinite neighbourhood of y, so O is infinite. Hence $R^{-1}(Y') \subset X'$. Inductively, one can prove that $R^{-1}(Y^\alpha) \subset X^\alpha$. This shows that $Y^\alpha \subset R(X^\alpha)$.

(*ii*) R is continuous if and only if R^{-1} is open, and the result follows from (*i*).

(*iii*) If R is a n-to-m correspondence from X to Y, then R^{-1} is a m-to-n correspondence from Y to X so it is sufficient to prove the second equality. We may assume that the degree of Y is an integer d. In that case, the rank of Y is a successor ordinal, say $\alpha + 1$. By the previous points, X also has rank $\alpha + 1$. Let us suppose for a contradiction that there be $O_0, \ldots, O_{d \cdot n}$ a sequence of $d \cdot n + 1$ open sets in X with maximal rank, and small intersections. The sets O_i^α are disjoint. As R is n-to-something, for every subset I of $[0, d \cdot n]$ having at least $n + 1$

points, the intersection $\bigcap_{i\in I} R(O_i^\alpha)$ is empty, so there exist $d+1$ disjoint subsets $I_0,\ldots,I_d$ of $[0,d\cdot n]$ such that for all j, the set $\bigcap_{i\in I_j} R(O_i^\alpha)$ is nonempty, and I_j is maximal with this property. Let us write V_j for $\bigcap_{i\in I_j} R(O_i)$. Every V_j is an open set in Y, with the same rank as Y by point (ii), and $V_j\cap V_k$ has small rank for $k\neq j$ in $[0,d]$, a contradiction with Y having degree d. □

Remark. In Model Theory, Cantor-Bendixson rank gave birth to Morley rank in omega-stable theories [3]. Points (i) and (ii) are well known by logicians and indicate that finite-to-one definable maps are "valuable" arrows for preserving a good notion of dimension [4].

Let X be a set and f a map on 2^X also defined on $2^X\times 2^X$. We say that f is *multiplicative* if $f(A\times B)$ equals $f(A)\times f(B)$. We call f a *pre-derivation* if $f(A\times B)$ equals $f(A)\times B\cup A\times f(B)$. Note a duality between some multiplicative maps and pre-derivations :

Lemma 2. *Let X be a set, and f a map from 2^X to 2^X such that $f(A)\subset A$ for every subset A of X. Write $\bar{f}(A)$ for $A\setminus f(A)$. Suppose in addition that f is defined on finite cartesian products of 2^X, and that f is multiplicative. Then $\bar{f}$ is a pre-derivation.*

For two spaces X and Y, we shall write $X\simeq Y$ if there is a finite correspondence from X to Y preserving the Cantor rank of each point. This is an equivalence relation.

We denote by $X\coprod Y$ the *topological disjoint union* of X and Y, that is, their disjoint union together with the finest topology for which the canonical injections $X\to X\coprod Y$ and $Y\to X\coprod Y$ are continuous.

Recall that any ordinal can be uniquely written as $\omega^{\alpha_1}.n_1+\ldots+\omega^{\alpha_k}.n_k$ where $\alpha_1,\ldots,\alpha_k$ is a strictly decreasing chain of ordinals and $n_1,\ldots,n_k$ are non-zero integers. This is known as its *Cantor normal form*. If α and β are two ordinals with normal forms $\omega^{\alpha_1}.m_1+\ldots+\omega^{\alpha_k}.m_k$ and $\omega^{\alpha_1}.n_1+\ldots+\omega^{\alpha_k}.n_k$ respectively (with zero integers possibly to make their length match), their *Cantor sum* $\alpha\oplus\beta$ is the ordinal $\omega^{\alpha_1}.(m_1+n_1)+\ldots+\omega^{\alpha_k}.(m_k+n_k)$.

Proposition 3. *The class of topological spaces modulo $\simeq$, together with $\coprod$ and $\times$ is a commutative integral semi-ring, on which Cantor's derivative is a derivation. The Cantor-Bendixson rank is a homomorphism from the class of compact spaces modulo $\simeq$ to the ordinals, preserving the structure of ordered semi-ring. Here, the ordinals are considered with the operations max, $\oplus$, and their natural ordering.*

Proof. It is not difficult to check that $\coprod$ and $\times$ survive modulo $\simeq$, are still associative, and that $\times$ is still distributive over $\coprod$. Note that if X and Y are two closed sub-spaces of some topological space Z, then $(X \cup Y)' = X' \cup Y'$. It follows that for any pair X and Y of topological spaces, $(X \coprod Y)'$ is homeomorphic to $X' \coprod Y'$, so Cantor's derivative preserves the sum $\coprod$ modulo $\simeq$. If we write $X^{isol.}$ for the set of isolated points in X, note that $(X \times Y)^{isol.}$ equals $X^{isol.} \times Y^{isol.}$. So Cantor's derivative is a pre-derivation by Lemma 2. The canonical map $f : X' \times Y \coprod X \times Y' \to X' \times Y \cup X \times Y'$ is a two-to-one continuous correspondence, so $f(x)$ has greater rank that x for every x by Lemma 1. For the converse, as $X' \times Y$ and $X \times Y'$ are both closed in $X \times Y$, one has $(X' \times Y \cup X \times Y')' = (X' \times Y)' \cup (X \times Y')'$, so f preserves the rank.

Cantor-Bendixson rank is well defined on the class of a topological space modulo $\simeq$. Clearly, the rank of a sum equals the maximum of the ranks. By induction, for any topological spaces X and Y, we get $(X \times Y)^\alpha = \cup_{\beta \oplus \gamma = \alpha} A^\beta \times B^\gamma$. Note that if X is a compact space, $CB(X)$ is a successor ordinal, the predecessor of which we write $CB^*(X)$. For two compact spaces X and Y, this shows that $CB^*(X \times Y)$ equals $CB^*(X) \oplus CB^*(Y)$. □

Lemma 1 implies that two spaces in finite continuous open correspondence have the same Cantor rank. Reciprocally, this invariant classifies countable Hausdorff locally compact spaces up to finite continuous open correspondances. This a consequence of the following :

Theorem 4 (Mazurkiewicz-Sierpiński [2])**.** *Every countable compact Hausdorff space is homeomorphic to some well-ordered set with the order topology.*

Proof. We give a short proof of a slightly more general result : we show that two countable locally compact Hausdorff spaces X and Y of same Cantor-Bendixson rank and degree are homeomorphic (in particular homeomorphic to $\omega^\alpha.d+1$ if they are compact of rank $\alpha+1$ and degree d).

Suppose first that X and Y be compact of rank $\alpha + 1$. Note that they are the disjoint union of finitely many compact spaces of degree 1, so one may assume that their degree is 1. We build a homeomorphism from X to Y by induction on the rank. Let $X_1, X_2, \dots$ and $Y_1, Y_2, \dots$ be two sequences of clopen sets roughly partitioning $X \setminus X^\alpha$ and $Y \setminus Y^\alpha$ respectively. As X_1 has smaller rank or degree than some finite union of Y_i, we may assume that X_1 has smaller rank or degree than Y_1, and

that Y_1 has smaller rank or degree that X_2 etc. We then build a back and forth : by induction hypothesis, there is sequence $f_1, g_1^{-1}, f_2, g_2^{-1} \dots$ of homeomorphism respectively from X_1 to some clopen $\tilde{Y}_1 \subset Y_1$, from $Y_1 \setminus \tilde{Y}_1$ to some clopen set $\tilde{X}_2 \subset X_2$, from $X_2 \setminus \tilde{X}_2$ to $\tilde{Y}_3 \subset Y_3$ etc. We call f be the union of all f_i and g_i, union one more map f_ω from X^α to Y^α and show that f is continuous. We may show sequential continuity as the spaces are metrisable. If x_i is a sequence of limit x in X, either x has small rank and belongs to some X_j, so $f(x_i)$ has limit $f(x)$ by continuity of f_j and g_j. Or x has maximal rank. If b is an accumulation point of the sequence $f(x_i)$ having small rank, it belongs to some clopen set Y_j, so the compact set X_j contains infinitely many x_i, a contradiction. So the sequence $f(x_i)$ has only one accumulation point and must converge to y.

If the spaces are locally compact, one can write them as a countable union of increasing clopen compact spaces, and builds a back and forth similarly. □

References

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